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ON HOPF ALGBERAS AND THE QUANTUM DOUBLE

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Abstract

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1 Introduction

Hopf Algebras, given their inextricable connection to the study of symmetry, have, for the last thirty years, regained a central stage in pure mathematics. This might come without surprise to any reader acquainted with the subtle and yet seemingly unfathomable prevalence that principles of symmetry have over any other when applied to our quest of understanding Nature's most intricate laws. Hopf algebras and their module-related theory some of which will be explored at length throughout the text, offer to mathematicians,

theoretical/condensed matter physicists or computer scientists some additional, hopefully more refined, lexicon for inspecting and communicating with greater clarity what is it that underpins the more fundamental ideas in their respective fields.

We will be mostly concerned with the application of these ideas to topics of condensed matter systems by means of relating the quantum double construction to certain topological quantum systems with non-abelian anyonic excitations and to implications for the project of quantization via connections of Hopf algebras (quantum groups really) to non-commutative geometry.

The approach followed in this project is inspired by the literature that has been produced over the last fifty years on Hopf algebras, quantum groups and their connections with different areas of physics. The true goal of the project is ultimately to be a useful prelude to a more dignified and extensive account of ways to apply the structures built here in the form of a master's thesis.

The next chapter will be boring unfortunately. It is mainly aimed at introducing through often vaguely looking and at times unintuitive definitions and fundamental results that will lend us a useful working characterization of Hopf Algebras. As inevitable as it is to go through them, hopefully, through the creeks of our slightly cumbersome definitions and notation, close examination of concrete examples will provide our intuitions a guiding torch.

2 Monoidal Categories

We begin this exposition by introducing the appropriate categorical context within which the modern connections between physics and mathematics are materialized. There are endless ways to introduce and motivate how the study of monoidal categories came to be, namely by referring the structures it attempts to generalize, these will hopefully be clear once we begin exploring concrete instances.

We assume some basic knowledge of category theory, far from any advanced level, hence making the section perfectly accessible for the undergraduate with internet connection and a working knowledge of Wikipedia.

The study of monoidal categories can, to some degree, be understood from the same point of view that originally lead to the need for categories: to better comprehend the interplay of quite distinct "mathematical theories", and doing so by finding in them the common feature of being built, essentially, by injecting sets with seemingly useful properties and studying functions between these (Set,Structure) pairs capable of translating the structure from the domain with the structure in the codomain. The same ethos inspires the definition of a monoidal category. Recall that a category, without any added statements about it, can be an awfully general object and hence a painful "thing" to work with, pointing us toward the need of better understanding at least some of the simplest constructions .

3 Algebras

For many readers, perhaps already familiar with these notions, this is either a dispensable portion of the text, a fresh and welcome recollection of ideas or an entirely new world. I can't afford to promise the latter a pristine and clear mapping of this new world and will instead present only a sufficiently useful summary.

Definition 3.1. An algebra over some fixed field k is a triple (A, m, η) consisting of a vector space A over a field k and two k-linear maps:

$$m\colon A \otimes A \to A \tag{1}$$

we call m the product or multiplication, we call η the unit map and they we require they satisfy the following axioms:

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \tag{2}$$
$$m \circ (\eta \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes \eta) = id \tag{3}$$

Equation (2) can, and should, be rephrased, in a seemingly less abstract tone, as the associativity of the product $ab := m(a \otimes b)$. Defined as:

$$(ab)c = a(bc) \qquad \forall a, b, c \in A$$

and equation (3) as the existence of a unit element $1_A := \eta(1)$ for the multiplication, more commonly defined as the element satisfying:

$$1_A a = a 1_A = a \qquad \forall a \in A$$

the fact that m is a k-linear map over $A \otimes A$ implies that this product is compatible, in a natural sense, with the inherent vector space structure on A.

Figure 1. provides a diagrammatic version of the axioms for an algebra over k. It might initially seem an unnecessary artifice to have them framed in this fashion, although as we will see they provide the perfect gateway to understand who is its symmetric partner, and our next object of inspection, the coalgebras over k. In fact if you reverse all arrows in that pair of diagrams and relabel those maps with something else, besides the identity ones, you get the axioms for the (co)-version of an algebra. At first glance this seems like an odd idea and one whose purpose is not immediately conveyed.

Before going any further with coalgebras we should first introduce the morphisms, or structure-preserving maps, between the objects central to this chapter. We call these algebra homomorphisms.

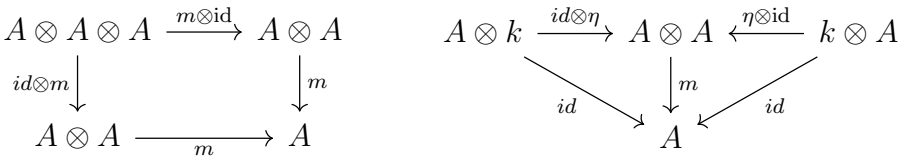


Figure 1: Axioms for an algebra (A, m, η) where commutativity of these diagrams is an equivalent formulation. Recall that for all vector spaces $A \cong A \otimes k$ are naturally identified by $\lambda \otimes v \mapsto \lambda v$.

Definition 3.2. A k-linear map $\varphi\colon A \longrightarrow B$ between two algebras[1] (A, m_A, η_A) and (B, m_B, η_B) is said to be an algebra morphism if:

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad \forall a, b \in A, \tag{4}$$
$$\varphi(1_A) = 1_B \tag{5}$$

Conditions (6) and (7) are very familiar if you have had any sort of introductory algebra course and they certainly suffice in providing the correct definition of algebra homomorphism but again we will turn to a less familiar but more conducive to abstract work. Conditions (6) and (7) can be given be rewritten via the following:

$$\varphi \circ m_A = m_B \circ (\varphi \otimes \varphi)$$
$$\varphi \circ \eta_A = \eta_B$$

Before we begin providing examples we should first note that the vector space $A \otimes B$ inherits from its factors a natural algebra structure given by the product $(a \otimes b)(a' \otimes b') := (ab \otimes b'b')$ corresponding to the following multiplication map:

$$m_{A\otimes B} = (m_A \otimes m_B) \circ (id \circ \tau \circ \mathrm{id})$$

the definition of the unit map $\eta_{A\otimes B}$ should be easily follow from our construction.

When we consider algebras rather than just vector spaces we have to face the fact that not all vector spaces are born equal. If the vector subspace $I \subseteq A$ also satisfies:

$$m(I \otimes A) + m(A \otimes I) \subseteq I$$

we say that I is a two-sided ideal of A. It would be equivalent to say that $aI \in I$ and that $Ia \in I$ for all $a \in A$. These are subspaces of particular importance given that they allow us to construct quotient algebras like A/I and thus providing us with a fruitful method for obtaining novel examples of algebras. The most typical way this creation process goes is by considering an ideal I generated by the elements we want to be considered as 0 in our new structure.

Let us no list some examples of algebras over a field k:

[1] $v \otimes w \xrightarrow{\tau} w \otimes v$ we will call τ the flip map

- The first and simplest example is to consider what we'll call the **opposite** algebra $(A^{op}, m_A \circ \tau, \eta_A)$. It looks exactly like the mirror image of the original algebra, $a \cdot_{op} b := b \cdot a$ and we will say that A is commutative iff $a \cdot_{op} b = a \cdot b$;

- The algebra of $\mathrm{M}_n(k)$ of $n \times n$ matrices with coefficients in k for the usual operations;

- k itself can be considered as an algebra with usual multiplication and is an example of a commutative algebra;

- The polynomial algebra $k[x]$ with its usual operations is a commutative algebra;

- The algebra $k[x, x^{-1}]$ of Laurent poynomials;

- The algebra $\mathrm{End}(V)$ of k-linear map from a vector space V to itself is an algebra for pointwise addition and composition - it is non-commutative for spaces of $\dim(V) > 1$;

- The tensor algebra $T(V)$ over a vector space is also one of the most useful starting points to generate important examples. We define $T(V) := \oplus_{k=0}^{\infty} V^{k\otimes}$ where by $V^{k\otimes} := V \otimes \cdots \otimes V$ the k-fold tensor product of V with itself. The vector space structure is evident, the multiplication is given by concatenation: if $v_n w_k := v_n \otimes w_k$ for $v_n \in V^{n\otimes}$, $w_k \in V^{k\otimes}$ and $v_n w_k \in V^{(n+k)\otimes}$. Notice it is a non-commutative algebra for $\dim(V) > 1$;

- Using the last example we construct two useful examples of algebras, the symmetric algebra and the exterior algebra. The symmetric algebra is given by considering the quotient algebra $Sym(V) := T(V)/I_+(V)$ where $I_+(V)$ is the ideal generated by elements of the form $x \otimes y - y \otimes x$ for $x, y \in T(V)$ and the exterior algebra $\bigwedge(V)$ is the equivalent construction for an ideal $I_-(V)$ generated by the elements $x \otimes x$ with $x \in T(V)$. In the case of the exterior algebra the product is then determined by $\pi(a \otimes b) = a \wedge b$, where $\pi \colon T(V) \longrightarrow \bigwedge(V)$ is the canonical projection. Some of the properties of the alternating product like $a \wedge a = 0$ or $a \wedge b = -b \wedge a$ will be familiar to some readers ;

- The group algebra kG of a finite group G is the vector space spanned by a basis $\{e_g : g \in G\}$ over some field k. The multiplication is defined by:

$$\Big(\sum_{g\in G} a(g)e_g\Big) \cdot \Big(\sum_{h\in G} b(h)e_h\Big) := \sum_{g,h\in G} a(g)b(h)e_{gh}$$

  where the coefficients $a(g), b(h) \in k$ and the unit is given by $\eta(1) := e_1$ where 1 is the identity of the group. This algebra is commutative iff the group is commutative;

- A slightly less common example is the algebra $U_q(b^+)$ defined as the algebra generated by the unit element 1 and the X, g, g^{-1}, quotiented by the ideal generated by the elements $gg^{-1} - 1$, $g^{-1}g - 1$ and $gX - qXg$ where we fix some non-zero scalar $q \in k$. Its importance will be revealed in due time, first let us note that it is an example of a non-commutative algebra.

3.1 Free Algebras

As with any set with which we endow particular structures, we can always the question: what is the least restricted object in the category of these structures that contains a given set X? By least restricted, or free, we mean that no additional relation is satisfied other than ones strictly imposed by its axioms.

In the category of algebras over some fixed field K, lets call it $\mathrm{Alg_k}$, we call this free object associated to some set X, the free algebra $k\{X\}$ (it is also written as $k\langle X\rangle$ in some books). The first and most explicit way to define it is by saying it consists of k-linear combinations of ordered concatenations of elements of X. By concatenations the reader should think of words like $x_{i_1}x_{i_2}\cdots x_{i_m}$ for $m \in \mathbb{N}$ and $x_{i_j} \in X$ generated by an alphabet X. This means a basis for this space is the set of all possible words for X. Two words are multiplied by $(x_{i_1}x_{i_2}\cdots x_{i_m})(x_{i_{m+1}}x_{k+2}\cdots x_{i_n}) = x_{i_1}x_{i_2}\cdots x_{i_m}x_{i_{m+1}}x_{k+2}\cdots x_{i_n}$ and the unit $1 = \varnothing$ is the empty word. An equivalent way to characterize this object, the proof of which one can find in, is to say that kX is the algebra for which, given any $A \in \mathrm{Alg_k}$ and any set-theoretic function $\varphi\colon X \longrightarrow A$ there exists a unique algebra morphism $\hat{\varphi}\colon k\{X\} \longrightarrow A$ such that $\hat{\varphi}(x) = \varphi(x)$. A more enlightened way to say this is:

$$\mathrm{Hom}_{\mathrm{Alg}}(k\{X\}, A) \cong \mathrm{Hom}_{\mathrm{Set}}(X, A)$$

where this is an isomorphism in the category of sets i.e., it is just a bijection. For a finite set $X = \{x_1, \ldots, x_n\}$ we can identify $\mathrm{Hom}_{\mathrm{Set}}(k\{x_1, \ldots, x_n\}, A) \equiv A^n$.

3.2 Affine Line and Affine Plane

If we want to work exclusively with commutative algebras, as our intuition dictates when building our archetypal lines, planes and their multidimensional counterparts, we need to restrict our free algebra object to reflect this. The way we do this is by quotienting $k\{X\}$ with the ideal I generated by the elements $x_ix_j - x_jx_i$. Now we are working over a very similar setting as before but we can commute any pair of elements in our words. This is clearly a consequence of the fact that we most certainly idealize the geometrical objects we are trying to model as being *constructed out of numbers*. If we were trying to model language this would be too tight a restriction since we wouldn't be able to distinguish the words 'MATE' from 'TEAM' (and what a pickle would we be in).

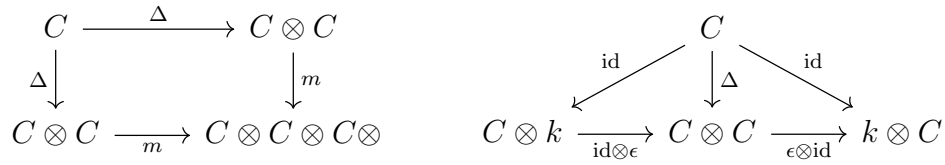


Figure 2: Axioms for a coalgebra (C, Δ, ϵ). Commutativity of the diagrams is an equivalent formulation to the axiomatic one.

Returning to $k\{X\}/I := k[X] = [x_1, \ldots, x_n]$ for some finite set X which we will call the polynomial algebra, we now have:

$$\mathrm{Hom}_{\mathrm{Alg}}(k[x_1, \ldots, x_n], A) = \{(a_1, \ldots, a_n\} \in A^n : a_i a_j = a_j a_i\}$$

which for $n = 1$ we call $k[x]$ the affine line and we call $\mathrm{Hom}_{\mathrm{Alg}}(k[x_1, \ldots, x_n], A) \cong A$ the A-points of the affine line. The same procedure applies for the case $n = 2$ for which we define the affine plane as $k[x, y]$ and the elements of $\mathrm{Hom}_{\mathrm{Alg}}(k[x, y], A) \cong A^2$ as A-points of the affine plane.

4 Coalgebras

If we take Figure 1. and reverse its arrows we get the diagrammatic version of the axioms for a coalgebra. At some point we will look at at these axioms from a more pragmatic standpoint when we begin studying representations or modules of these objects. For now lets right a proper definition:

Definition 4.1. A coalgebra over some field k is a triple (C, Δ, ϵ) consisting of a vector space C over a field k and a pair of k-linear maps:

$$\Delta \colon C \otimes C \longrightarrow C \qquad \epsilon \colon k \longrightarrow C$$

we say that Δ is acoproduct or comultiplication, we call ϵ the counit map and they require they satisfy the following axioms:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (id \otimes \Delta) \circ \Delta \tag{6}$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \eta) \circ \Delta = \mathrm{id} \tag{7}$$

equation (6) is called the coassociative property of Δ and (7) is usually called the counital axioms. The diagrams below in Figure 2. capture the dual nature of our definition when compared to Figure 1.

Before providing any more intuition for these objects we should first introduce the (co) counterparts to the notions of an algebra homomorphism and two-sided ideal.

Definition 4.2. A coalgebra homorphism $\psi\colon C \longrightarrow D$ between coalgberas $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ is a k-linear map satisfying:

$$(\psi \otimes \psi) \circ \Delta_C = \Delta_D \circ \psi \tag{8}$$
$$\epsilon_D \circ \psi = \epsilon_C \tag{9}$$

The same way we defined a natural algebra structure on the tensor product of two algebras, we can define a natural structure of coalgebra for C by defining the following comultiplication and counit:

$$\Delta_C := (id \otimes \tau \otimes \mathrm{id}) \circ (\Delta_A \otimes \Delta_D)$$
$$\epsilon_C := \epsilon_C \otimes \epsilon_D$$

we can also define a notion of cocommutativity by requiring that the cooposite coalgebra is equal to its coalgebra. In simple terms, for a coalgebra to be cocommutative $\Delta^{op} := \tau \circ \Delta = \Delta$. To end the mimic of definitions from the algebra chapter we say that $I \subseteq C$ is two-sided coideal of C if:

$$\Delta(I) \subseteq I \otimes C + C \otimes I \qquad \epsilon(I) = 0$$

and the quotient C/I has a coalgebra structure induced by the projection homomorphism.

In order to deal with the less intuitive notion of comultiplication we will introduce the so-called Sweedler notation. If $c \in C$ is an element of the coalgebra, then $\Delta(c)$ is a finite sum of terms in $C \otimes C$:

$$\Delta(c) = \sum_i c_{i1} \otimes c_{i2} \qquad c_{i1}, c_{i2} \in C \tag{10}$$

moreover, this representation isn't necessarily unique, hence for notational simplicity we will write only $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. It works because it still conveys the fact that it is a sum of ordered tensor factors while not necessarily making a choice for a representation. To get a sense of the power the notation offers lets see how it is particularly well handled by coassociativity:

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\big[\sum c_{(1)} \otimes c_{(2)}\big] = (id \otimes \Delta)\big[\sum c_{(1)} \otimes c_{(2)}\big] \qquad &\Leftrightarrow \\
\sum \Delta(c_{(1)}) \otimes c_{(2)} = \sum c_{(1)} \otimes \Delta(c_{(2)}) \qquad &\Leftrightarrow \\
\sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} &
\end{aligned}$$

This property of the notation means that we can just write $\Delta^{(2)}(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. The counit axiom can be wrtitten as:

$$\sum \epsilon(c_{(1)})c_{(2)} = \sum c_{(1)}\epsilon(c_{(2)}) = c, \qquad \forall c \in C \tag{11}$$

Now we will begin introducing the structures that bring coalgebras and algebras into harmony: the bialgebras and Hopf algberas. The next proposition will help us build some important inuition:

Proposition 4.1. *Let A be a coalgebra and B be an algebra then the vector space ${}^{lin}_{k}(A, B)$ of k-linear maps between them is an algebra for the following so-called convolution product:*

$$(\varphi * \psi)(a) := m_B \circ (\varphi \otimes \psi) \circ \Delta_A(a) \equiv \sum f(a_{(1)})g(a_{(2)}) \qquad \forall a \in A \tag{12}$$

and unit given by $\eta_B \circ \epsilon_A$.

Proof. The associativity of the convolution comes as a direct computational consequence of the associativiy of m_B and the coassociativity of Δ_A:

$$((\varphi * \psi) * \chi)(a) = \sum (\varphi * \psi)(a_{(1)})\chi(a_{(2)}) = \sum \varphi(a_{(1)})\psi(a_{(2)})\chi(a_{(3)}) = (\varphi * (\psi * \chi))(a)$$

we are then left only with task of showing that $\eta_B \circ \epsilon_A$ is in fact a unit. The following computation provides the anwser:

$$((\eta_B \circ \epsilon_A) * \varphi)(a) = \sum \epsilon_A(a_{(1)})\varphi(a_{(2)}) = \varphi\Big(\sum \epsilon_A(a_{(1)})a_{(2)}\Big) = \varphi(a)$$

establishing it as also a right unit is done in mirror image of the last equation. □

The most immediate, and extremely important result, to follow is that the dual vector space A^* can be given an algebra structure.

Corollary 1. *The dual vector space A^* of a coalgebra A is an algebra for the product given by:*

$$(\alpha\beta)(a) := \sum \alpha(a_{(1)})\beta(a_{(2)}, \qquad \forall a \in A, \qquad \forall \alpha, \beta \in A^* \tag{13}$$

The same operation can't be done verbatim for algebras if they're infinite-dimensional because $\Delta_{B^*}(\alpha) := \alpha \circ m_B \colon B^* \longrightarrow (B \otimes B)^*$ might not be an element of $B^* \otimes B^*$. The way this issue is typically confronted is by either limiting our coproduct to a subspace of the dual or by performing a topological completion of the tensor product by enlarging it. In this text we will stick to the finite-dimensional case so the use of these tools will be entirely avoided.

Example 4.1. We can associate to any set X, via the free functor, a vector space $k[X] = \oplus_{x \in X} k$ with basis elements given by $x \in X$. We can add a natural coalgebra structure on $k[X]$ given by:

$$\Delta(x) = x \otimes x, \qquad \epsilon(x) = 1, \qquad \forall x \in X$$

and extended linearly. We can think of Δ as linear duplication map and see that:

$$(\Delta \otimes \mathrm{id})\Delta(x) = (x \otimes x) \otimes x = x \otimes (x \otimes x) = (\mathrm{id} \otimes \Delta)\Delta(x)$$
$$(\epsilon \otimes \mathrm{id})\Delta(x) = \epsilon(x)x = x = x\epsilon(x) = (\mathrm{id} \otimes \epsilon)\Delta(x)$$

in line with the last proposition we can construct the algebra $k[X]^*$ of linear forms or dual vectors $\alpha : k[X] \longrightarrow k$. With product given by:

$$(\alpha\beta)(x) := \sum \alpha(a_{(1)})\beta(a_{(2)}) = \alpha(x)\beta(x)$$

more will be said later on about this construction.

We postpone the exploration of additional examples of coalgebras for when we can appreciate their structure naturally interacting with our more familiar notion of algebra, but let it be said that they are as naturally occurring in mathematics as algebras are, they are just not as emphatically spoken about or in most cases just not labelled and categorized as a coalgebra object.

5 Bialgebras

Simply put, a bialgebra is a vector space with an algebra structure and a coalgebra structure that are compatible. The motivation for this compatibility emerges abundantly clear from the following result:

Proposition 5.1. *If a vector space A is given both an algebra and a coalgebra structure, then the following statements are equivalent:*

1. *The comultiplication $\Delta_A \colon A \longrightarrow A \otimes A$ and counit $\epsilon \colon A \longrightarrow k$ are algebra homomorphisms;*

2. *The multiplication $m_A \colon A \otimes A \longrightarrow A$ and unit $\eta \colon k \longrightarrow A$ are coalgebra homomorphisms;*

Proof. Let us restate both statements respectively: Perhaps the most elucidating way of exposing the equivalence of our statements is to note that statements (1) can be reformulated as the pair of the the following commutative diagrams:

□

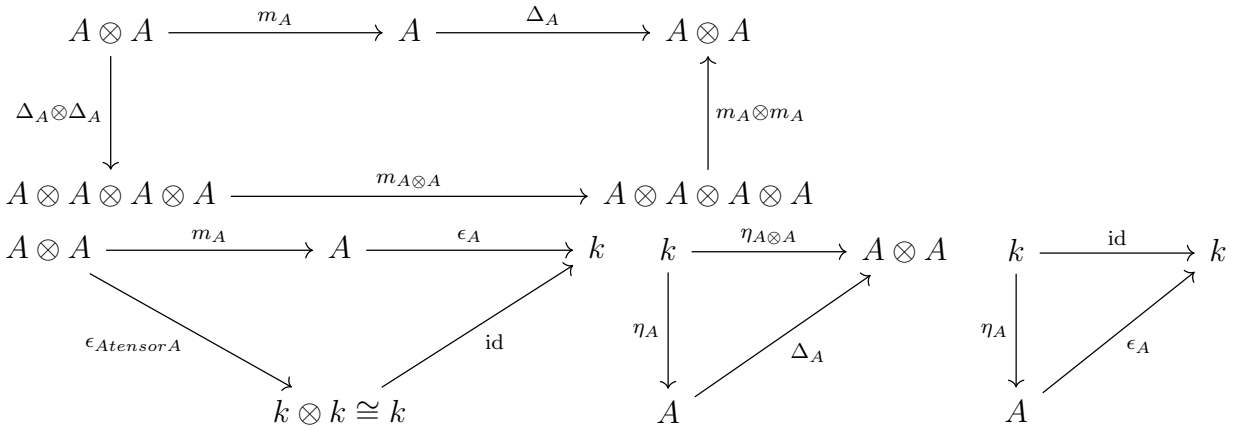


Figure 3: Commutative diagrams for Δ and ϵ to be algebra homomorphisms

Definition 5.1. A bialgebra A is an algebra and a coalgebra that satisfy either one, and hence both, of the conditions stated in the last proposition. Equivalently they must satisfy the following conditions for all $a, b \in A$:

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \Delta(1) = 1 \otimes 1, \qquad \epsilon(1) = 1$$

where we use 1 to refer to the algebra identity element when its clear from context to which algebra it belongs.

Now the same procedural steps that followed our definition of both algebra and coalgebra structure could be applied to the bialgebras, namely we can define an appropriate homomorphism between a pair of bialgebras and we can also define a two-sided biideal I by asking that the subspace I be both an ideal and a coideal for its respective algbera and coalgebra structures. The tensor product $A \otimes B$ of two bialgebras can also be made into a bialgebra by applying our previous procedures. The novel aspect now in that we can consider three possible opposite bialgebra structures, we can consider its opposite algebra structure and retain its coalgebra structure which we will denote as A^{op}, the case A^{cop} we consider its (co)opposite coalgebra structure and retain its original algebra structure and the third and last case which denote as $A^{op,cop}$ where we consider both the opposite and (co)opposite algebra and coalgebra operations. This reflects the fact that a bialgebra can be:

- commutative & cocommutative,
- non-commutative & cocommutative,
- commutative & non-cocommutative,
- non-commutative & non-cocommutative.

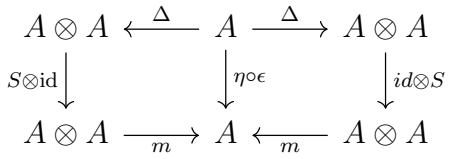


Figure 4: Hopf algbera axioms

6 Hopf Algebras

We finally come to the much anticipated notion of a Hopf Algebra.

Definition 6.1. A Hopf algebra A is a bialgebra together with a k-linear map $S\colon A \longrightarrow A$ called the antipode satisfying:

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \eta \circ \epsilon \tag{14}$$

in effect making the diagram in Figure 4. commute.

in Sweedler notation we can reproduce these equalities as:

$$\sum S(a_{(1)})a_{(2)} = \sum a_{(1)}S(a_{(2)}) = \eta(\epsilon(a)) = \epsilon(a)1 = \epsilon(a)$$

If we now employ our results about the algebra $({}^{lin}_{k}(A, A), *, \eta \circ \epsilon)$, we can see that $S * \mathrm{id} = id * S = \eta \circ \epsilon$, in other words the antipode S plays the role of inverse (in the convolution product) to the identity $\mathrm{id} \in^{lin}_{k} (A, A)$. Recall that the unit of this algebra is given by the element $\eta \circ \epsilon$. The added advantage of this interpretation of the antipode S is that it immediately establishes its uniqueness within a bialgebra by the familiar, and simple to prove, uniqueness of the inverse element in a monoid.

Let us now state some useful properties about the antipode and about Hopf algebras in general that are gonna turn out to be quite useful in the process of building interesting examples.

Proposition 6.1. *The antipode S for a Hopf algebra mcA is an algebra homomorphism $S\colon A \longrightarrow A^{op}$ and also $S\colon A \longrightarrow A^{cop}$ a coalgebra homomorphism. These are also called algebra and coalgebra antihomomorphisms respectively. An equivalent way to say this is to say that S satisfies the following equations for all $a, b \in A$:*

$$S(ab) = S(a)S(b), \qquad S(1) = 1 \tag{15}$$

$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta, \qquad \epsilon \circ S = \epsilon \tag{16}$$

Proof. We will only prove equation (18), the complete proof can be found in ?????, and is a great exercise in efficiently applying the bialgebra axioms together coupled with the

clever usage of Sweedler notation.

$$\begin{aligned}
S(b)S(a) &= \sum S(b_{(1)}\epsilon(b_{(2)}))S(a_{(1)}\epsilon(a_{(2)})) \\
&= \sum S(b_{(1)})S(a_{(1)})\epsilon(b_{(2)}a_{(2)}) \\
&= \sum S(b_{(1)})S(a_{(1)})(a_{(2)}b_{(2)})_{(1)}S((a_{(2)}b_{(2)})_{(2)}) \\
&= \sum S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}S(a_{(3)}b_{(3)}) \\
&= \sum S(b_{(1)})(\epsilon(a_{(1)}1)b_{(2)}S(a_{(2)}b_{(3)}) \\
&= \sum \epsilon(a_{(1)})\epsilon(b_{(1)})S(a_{(2)}b_{(2)}) \\
&= \sum S(\epsilon(a_{(1)})a_{(2)}\epsilon(b_{(1)})b_{(2)}) \\
&= S(ab)
\end{aligned}$$

In the first two equalities we apply the antipode axiom plus the fact that S is a linear map and the the fact ϵ takes values in k, the third and forth are again the antipode axioms this time with the help of the coassociativity of Δ, the fifth and sixth are again th antipode axiom with the additional readjustment of the order of the factors and the the last ones are self-evident. We will usually write property (19) as $\sum S(a)_{(1)} \otimes S(a)_{(2)} = \sum S(a_{(2)}) \otimes S(a_{(1)})$. □

We can also check that if the bialgbera $(A, m, \Delta, \eta, \epsilon)$ is Hopf, then since:

$$\begin{aligned}
m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta^{cop} &= (m \circ \tau) \circ (S \otimes \mathrm{id}) \circ (\tau \circ \Delta) \\
&= m \circ (id \otimes S) \circ \tau \circ \Delta \\
&= m \circ (S \otimes \mathrm{id}) \circ \Delta \\
&= \eta \circ \epsilon
\end{aligned}$$

we get that $A^{op,cop}$ is also a Hopf algbera. For the opposite and coopposite algebras we need the invertibility S as a k-linear map.

Proposition 6.2. *Let $(A, m, \Delta, \eta, \epsilon, S)$ be a Hopf algebra then we have the following equivalent statements:*

1. *The antipode $S\colon A \longrightarrow A$ is invertible as a k-linear mapping;*
2. *The bialgebra A^{op} is a Hopf algebra;*
3. *The bialgebra A^{cop} is a Hopf algebra;*

and the map S^{-1} is the antipode of A^{op} and A^{cop}.

Proof. Let us prove $1 \Leftrightarrow 2$ since the case for 3 is essentially the same. Let us begin by proving $1 \Rightarrow 2$:

$$\begin{aligned}\sum S^{-1}(a_{(2)})a_{(1)} &= \sum S^{-1}(a_{(2)})S^{-1}(S(a_{(1)})) \\ &= S^{-1}\Big(\sum a_{(1)}S(a_{(2)})\Big) = S^{-1}(\epsilon(a)1) \\ &= \epsilon(a)1\end{aligned}$$

where we made clear use of the fact that S^{-1} is still an antialgebra homomorphism. Now we prove the other direction, and we assume that S' is A^{op}'s antipode:

$$\begin{aligned}S'S(a) &= S'S\Big(\sum \epsilon(a_{(2)})a_{(1)}\Big) \\ &= \sum \epsilon(a_{(1)})S'(S(a_{(1)})) \\ &= \sum a_{(3)}S'(a_{(2)})S'(S(a_{(1)})) \\ &= \sum a_{(3)}S'(a_{(1)}S(a_{(2)})) \\ &= \sum a_{(2)}S'(\epsilon(a_{(1)})1) \\ &= \sum a_{(2)}\epsilon(a_{(1)})S'(1) = a\end{aligned}$$

the same calculation works for SS' and hence we finish our proof. □

This leads to an important result:

Proposition 6.3. *If A is a commutative or cocommutative Hopf algebra then $S^2 =$.*

Proof. Just notice that if A is commutative we have:

$$\sum a_{(2)}S(a_{(1)}) = \sum S(a_{(1)})a_{(2)} = \epsilon(a)1$$

implying that S is an antipode for A^{op} and hence $S^{-1} = S$. For cocommutativity we have:

$$\sum a_{(2)}S(a_{(1)}) = \sum a_{(1)}S(a_{(2)}) = \epsilon(a)1$$

and again impying $S^2 =$. □

One last result worth mentioning before looking at examples is that given by the following proposition:

Proposition 6.4. *For an algebra A, it suffices to prove that the algebra homomorphisms Δ, ϵ and the algebra antihomomorphism S satisfy the coassociative and counit (or antipode axioms, in the case of a Hopf alegbra), respectively, for a generating subset of A.*

Example 6.1. The group algebra kG for a finite group G, discussed in the first chapter, is in fact also a Hopf algebra with following additional structure:

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1, \qquad S(g) = g^{-1}$$

We simplify notation here by referring to elements of kG as formal k-linear combinations of elements of G at no added risk of ambiguity. The associativity and coassociativity both come as expressions of the associativity of the multilplication for the basis elements. The counit offers no resistance and S satisfies: $S(gh) = (gh)^{-1} = h^{-1}g^{-1} = S(h)S(g)$ and $m \circ (S \otimes \mathrm{id}) \circ \Delta(g) = S(g)g = g^{-1}g = 1 = \epsilon(g)$ for all $g \in G \subseteq kG$, ensuring it is an antipode. This is a cocommutative Hopf algbera.

Example 6.2. The universal envelping algbera $U(\mathfrak{g})$ over some Lie algebra $\big(\mathfrak{g}, [\ ,\ ], k\big)$ can be also be given a Hopf algebra structure. We recall two useful ways of arrivning at this algebra: the first is as the tensor algebra $T(\mathfrak{g})$ modded out by the ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in T(\mathfrak{g})$. As an example the element $1 + x \otimes y \otimes z - y \otimes x \otimes z$ would be identified with the element $1 + [x, y] \otimes z$. As an alternative prescription we could instead define $U(\mathfrak{g})$ as the algebra satisfying the following universal property:

For all algebras A and algebra morphism $\varphi : \mathfrak{g}$ satisfying $\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ for all $x, y \in \mathfrak{g}$ there is a unique algebra homomorphism $\tilde{\varphi}\colon U(\mathfrak{g})$ and satisfies $\varphi(x) = \tilde{\varphi}(x)$ for all $x \in \mathfrak{g}$. This second approach proves crucial in establishing the legitimacy of the following maps as Hopf algbera maps:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \epsilon(x) = 0, \qquad S(x) = -x$$

If we interpret these as algebra maps to $A = U(\mathfrak{g}) \otimes U(\mathfrak{g}), k, U(\mathfrak{g})^{op}$ respectively, it is clear that both ϵ and S satisfy the bracket condition and hence extend to algebra homomorphisms from $U(\mathfrak{g})$ to A. For Δ we should explicitly check:

$$\begin{aligned}
\Delta(x)\Delta(y) - \Delta(y)Delt(x) &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x) \\
&= x \otimes y \otimes 1 + 1 \otimes x \otimes y + x \otimes y + y \otimes x \\
&\quad - y \otimes x \otimes 1 - 1 \otimes y \otimes x - x \otimes y - y \otimes x \\
&= [x, y] \otimes 1 - 1 \otimes [x, y] \\
&= \Delta([x, y])
\end{aligned}$$

so Δ too is an algebra homomorphism. We have now only to make sure that the counit,

coassociative and antipode axioms are satisfied for all $x \in \mathfrak{g}$. This is an easy thing to do:

$$\begin{aligned}(\Delta \otimes \mathrm{id})(\Delta(x)) &= (\Delta \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) \\ &= (x \otimes 1 + 1 \otimes x) \otimes 1 + (1 \otimes 1 + 1 \otimes 1) \otimes x \\ &= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x + 1 \otimes 1 \otimes x \\ &= x \otimes (1 \otimes 1 + 1 \otimes 1) + 1 \otimes (x \otimes 1 + 1 \otimes x) \\ &= (id \otimes \Delta)(\Delta(x)); \\ m(S \otimes \mathrm{id})(\Delta(x)) &= m(S \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) \\ &= m(-x \otimes 1 + 1 \otimes x) = -x \otimes 1 + 1 \otimes x = 0 = \epsilon(x)1\end{aligned}$$

all these hold for all $x \in \mathfrak{g}$. This is a cocommutative Hopf algebra.

The next structure is the first example of a noncocommutative and noncommutative Hopf algbera, it was first put forth by M.Sweedler and is the smallest dimensional complex Hopf algebra with those properties. It is a kind of mixture between both of the previous constructions:

Example 6.3. $U_q(b^+)$, we define the following additional structures:

$$\begin{gathered}\Delta(X) = X \otimes 1 + g \otimes X, \qquad \Delta(g) = g \otimes g, \qquad \Delta(g^{-1}) = g^{-1} \otimes g^{-1} \\ \epsilon(X) = 0, \qquad \epsilon(g) = 1, \qquad \epsilon(g^{-1}) = 1, \\ S(g) = g^{-1}, \qquad S(X) = -g^{-1}X, \qquad S(g^{-1}) = g\end{gathered}$$

By defining the maps on the generators of the algebra and extending them multiplicatively we are then just tasked with making sure the extensions respect the relations and hence ensuring Δ and ϵ do in fact descend to algebra homomorphisms between the quotient algebra we are working with and lastly that they satisfy the coassociative and counit axioms for $1, X, g, g^{-1}$. We will typically skip the latter and delegate the calculation to the more skeptic reader. We have, for instance:

$$\begin{aligned}\Delta(gX) = \Delta(g)\Delta(X) &= (g \otimes g)(X \otimes 1 + g \otimes X) \\ &= gX \otimes g + g^2 \otimes gX = qXg \otimes g + g^2 \otimes qXg \\ &= q((X \otimes 1)(g \otimes g) + (g \otimes X)(g \otimes g)) \\ &= q((X \otimes 1) + (g \otimes X))(g \otimes g) \\ &= q\Delta(X)\Delta(g) \\ &= \Delta(qXg)\end{aligned}$$

proving that Δ respects the quotient. For the counit and coproduct the rest of the relations are trivially checked. To establish $U_q(b^+)$ as a bonafide Hopf algebra We still need

to guarantee that S, extended multiplicatively as an algebra antihomomorphism, also respects the relations and satisfies the antipode axioms for the generators. We see that:

$$\begin{aligned} S(gX) = S(X)S(g) &= -g^{-1}Xg^{-1} \\ &= -g^{-1}g^{-1}gXg^{-1} \\ &= -g^{-1}g^{-1}qXgg^{-1} \\ &= -qg^{-1}g^{-1}X \\ &= qS(g)S(X) \\ &= S(qXg) \end{aligned}$$

once again all the other relations are easily seen to satisfy the antihomomorphism requirements.

Having already spent some time going through interesting examples the next definition and result should feel, at least in part, to be enriching our Hopf algebra lingo:

Definition 6.2. For a bialgbera A we say that a non-zero $g \in A$ is group-like if $\Delta(g) = g \otimes g$. If $x \in A$ satisfies $\Delta(x) = x \otimes 1 + 1 \otimes x$ it is said be primitive.

Proposition 6.5. *The product of group-like elements in a bialgebra A is also group-like. A primitive element $x \in A$ satisfies $\epsilon(x) = 0$ and $[x, y] := xy - yx$ is also a primitive element.*

Proof. If $g, h \in A$ are a pair of group-like elements then we have $\Delta(h) = h \otimes h$ and $\Delta(g) = g \otimes g$. Then since A is a bialgebra we have that $\Delta(gh) = \Delta(g)\Delta(h) = (g \otimes g)(h \otimes h) = gh \otimes gh$ which proves the first statement. The second can be split in two, lets prove the first:

$$\begin{aligned} (\epsilon \otimes \mathrm{id})(\Delta(x)) &= (\epsilon \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) \\ &= \epsilon(x)1 + x = x = \mathrm{id}(x) \Rightarrow \epsilon(x) = 0 \end{aligned}$$

now the second is precisely the calculation done for the universal enveloping algebras. □

7 Function Spaces over Groups

In this chapter we will employ some of the philosophy handed down to us by Grothendieck. We do it so by trying to understand a space, in our case a group G, by means of understanding the space of functions defined over the group. This, he thought, was how you could ultimately understand the connection between algebra and geometry. In some sense, the next paragraphs or this section in general, could have fulfilled a more introductory role for studying Hopf algebras.

Let us begin with a very general, albeit useful, approach. Let us begin by defining a simple functor:

Definition 7.1. Let $\mathcal{F} : \mathrm{Set}^{Fin} \longrightarrow \mathrm{ComAlg}_k^{Fin}$ be a contravariant functor from the category Set^{Fin} of finite sets to the category ComAlg^{Fin} of finite-dimensional commutative algebras which maps sets in the following way:

- $\mathcal{F}(X) = k(X) := \{f \colon X \longrightarrow k,\ f \text{ is a function}\} = \mathrm{Span}_k(\{\delta_x\}_{\in x})$;
- $\mathcal{F}(\varphi \colon X \longrightarrow Y) : k^Y \longrightarrow k(X)$ which maps $\mathcal{F}(\varphi)(f) = f \circ \varphi$

δ_x are the basis elements for $k(X)$ and are defined as $\delta_x(y) = \delta_{x,y}$ where the last delta is the Kronecker delta. We know that $k(X)$ is a commutative algebra for the maps :

$$\begin{aligned} \cdot \colon k(X) \times k(X) &\longrightarrow k(X) \\ (f, g) &\longmapsto (f \cdot g)(x) := f(x)g(x), \quad \forall x \in X \\ \eta \colon k &\longmapsto k(X) \\ 1 &\longmapsto 1_{k(X)}(x) := 1, \quad \forall x \in X \end{aligned}$$

because $\cdot$ is a k-bilinear map we can redefine it as a k-linear map $\cdot \colon A \otimes A \longrightarrow A$ and it is easy to check that they satisfy the associative and unital axioms. One interesting route to pursue, even thouh we will not, is to ponder whether this functor is a left or right adjoint to some other functor. The answer is yes, it is called the **Spec** functor and it of tremendous value for algebraic geometers. We will instead focus on a particular kind of combination of sets and maps (the title of section may give away which kind).

A finite group G can be thought of as a set G together with three kinds of maps:

$$\begin{aligned} m \colon G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \\ u \colon 1 &\longrightarrow G \\ 1 &\longmapsto e \\ s \colon G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

We can now ask ourselves what should be corresponding images of these morphisms when applying the functor $\mathcal{F}$? The contravariant nature of the functor should instantly take us to to some kinds of comultiplication and counit. We will now explore this in more detail, and a lot of what we will do will feel as if this interlude could have been dispensed - It could have, but in honesty it was meant to convey the abstract plant for the constructions we will be focusing on in this chapter.

Let us consider a finite group G. (The finite condition on G could have been weakened at the expense of being a lot more careful than we intend to be in this chapter). Consider the space $\mathcal{F}(G)$ of k-valued functions on G, it is an algebra for point-wise operations

as we have seen. Since G is a finite group we can establish then the following algebra isomorphism:

$$\begin{aligned}\mathcal{F}(G\times G) &\longrightarrow \mathcal{F}(G)\otimes\mathcal{F}(G)\\ \varphi &\longmapsto (\varphi\otimes\varphi)(g\otimes h) := \varphi(g,h)\end{aligned}$$

$$\begin{aligned}\mathcal{F}(G)\otimes\mathcal{F}(G) &\longrightarrow \mathcal{F}(G\times G)\\ \psi_1\otimes\psi_2 &\longmapsto \tilde{\psi}(g,h) := \psi_1(g)\psi_2(h)\end{aligned}$$

If G were not a finite group we could only really guarantee a k-algebra embedding: $\mathcal{F}(G)\otimes\mathcal{F}(G)\subseteq\mathcal{F}(G\times G)$. Having said this we can ask how what properties do the the maps $\Delta := \mathcal{F}(m)$, $\epsilon := \mathcal{F}(u)$ and $S := \mathcal{F}(s)$ have? Well, perhaps unsurprisingly, they are the missing ingredients to turn $\mathcal{F}(G)$ into a Hopf algebra. Let us explicitly write down what they do:

$$\begin{aligned}\Delta\colon \mathcal{F}(G) &\longrightarrow \mathcal{F}(G)\otimes\mathcal{F}(G)\\ \varphi &\longmapsto \Delta(\varphi)(g\otimes h)\cong\Delta(\varphi)(g,h) := \varphi(gh),\quad \forall\varphi\in\mathcal{F}(G),\forall g,h\in G\\ \epsilon : \mathcal{F}(G) &\longrightarrow \mathbb{C}\\ \varphi &\longmapsto \varphi(e),\quad \forall\varphi\in\mathcal{F}(G) \text{ and } e \text{ is the identity element of } G\\ S : \mathcal{F}(G) &\longrightarrow \mathcal{F}(G)\\ \varphi &\longmapsto S(\varphi)(g) := \varphi(g^{-1}),\quad \forall\varphi\in\mathcal{F}(G),\forall g\in G\end{aligned}$$

The properties satisfied by the group multiplication, unit and inverse, such as associativity, unit property and inverse property are now encoded as the coassociativite property, counit property and antipode axioms. Lets list all the operations on basis elements to simplify proving these properties:

- $\cdot(\delta_g\otimes\delta_h) = \delta_{gh}$;
- $\eta(1) = 1_{\mathcal{F}(G)} := \sum_{g\in G}\delta_g$;
- $\Delta(\delta_g) = \sum_{h\in G}\delta_h\otimes\delta_{h^{-1}g}$;
- $\epsilon(\delta_g) = \delta_{g,e}$;
- $S(\delta_g) = \delta_{g^{-1}}$

Lets verify only the bialgebra and antipode axioms:

$$\Delta(\varphi\cdot\psi)(gh) = \varphi\cdot\psi(gh) = \varphi(gh)\psi(gh) = [\Delta(\varphi)\cdot\Delta(\psi)](gh)$$

$$\Delta(1_{\mathcal{F}(G)}) = \sum_{g\in G}\Delta(\delta_g) = \sum_{g\in G}\sum_{h\in G}\delta_h\otimes\delta_{h^{-1}g} = \sum_{g\in G}\sum_{h\in G}\delta_g\otimes\delta_h = 1_{\mathcal{F}(G)}\otimes 1_{\mathcal{F}(G)}$$

we will skip the counit case and jump straight to the antipode:

$$\sum_{k\in G} S(\delta_k)\delta_{k^{-1}g} = \sum_{k\in G} \delta_{k^{-1}}\delta_{k^{-1}g} = \sum_{k\in G} \delta_{k^{-1},k^{-1}g}\delta_{k^{-1}} = \sum_{k\in G} \delta_{e,g}\delta_{k^{-1}} = \epsilon(\delta_g)\sum_{k\in G}\delta_k = \epsilon(\delta_g)1_{\mathcal{F}(G)}$$

the other verification is entirely analogous, and with it we finally establish $(\mathcal{F}(G), \cdot, \eta, \Delta, \epsilon, S)$ as a Hopf algebra over k. It is a commutative and not necessarily cocommutive example.

To cap off this chapter we will construct a more convoluted example. We will use the construction above and apply it to simple matrix Lie groups, without truly wrinkling anything, by restricting our functions to the subspace generated by the $\mathcal{O}(G) \subseteq k(G) = \mathcal{F}(G)$ of the coordinate functions $x^i_j : G \longrightarrow k$ ($\mathcal{O}(G) \cong k[x^i_j]$ the freely generated commutative algebra over k) defined for an element $g = (g)_{ij} \in G$ as $x^i_j(g) = g_{ij}$.

Lets look in detail at the case of $\mathcal{O}(SL_2)$. Remember from the beginning of the chapter, as this is still a space of functions over a group, how we build a coproduct a counit and an antipode from the group multiplication, unit and inverse respectively (in this case these are matrix multiplication, identity matrix and matrix inversion also respectively):

$$\Delta(x^i_j)(g,h) := x^i_j(gh) = (gh)_{ij} = \sum_k (g)_{ik}(h)_{kj} = \sum_k x^i_k(g)x^k_j(h) = \Big[\sum_k x^i_k \otimes x^k_j\Big](g\otimes h)$$

$$\Rightarrow\ \Delta(x^i_j) := \sum_k x^i_k \otimes x^k_j \quad \in \mathcal{O}(SL_2)\otimes\mathcal{O}(SL_2);$$

$$\epsilon(x^i_j) := x^i_j(I_{2\times 2}) = \delta^i_j \in \mathbb{C};$$

$$S(x^i_j)(g) := x^i_j(g^{-1}) = x^1_1(g)x^2_2(g) - x^1_2(g)x^2_1(g) = [x^1_1x^2_2 - x^1_2x^2_1](g)$$

$$\Rightarrow\ S(x^i_j) := x^1_1x^2_2 - x^1_2x^2_1 \quad \in \mathcal{O}(SL_2)$$

To simplify the rest of our discussion let us relabel these functions as:

$$x^1_1 = a, \qquad x^1_2 = b, \qquad x^2_1 = c, \qquad x^2_2 = d$$

with the counit then given by $\epsilon(a) = \epsilon(d) = 1$ and $\epsilon(b) = \epsilon(c) = 0$ We have the following comultiplication rules:

$$\begin{aligned} \Delta(a) &= a\otimes a + b\otimes c, & \Delta(b) &= a\otimes b + b\otimes d \\ \Delta(c) &= c\otimes a + d\otimes c, & \Delta(d) &= c\otimes b + d\otimes d \end{aligned}$$

motivating the following way to denote the coproduct:

$$\Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix} := \begin{pmatrix} a & b \\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d\end{pmatrix}$$

the counit and antipode can also be given in this reduced form:

$$\begin{pmatrix}\epsilon(a) & \epsilon(b)\\ \epsilon(c) & \epsilon(d)\end{pmatrix} = \epsilon\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}; \quad \begin{pmatrix}S(a) & S(b)\\ S(c) & S(d)\end{pmatrix} = S\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}$$

Lets check the antipode axioms for all generators at once by making use of this notation:

$$S\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot S\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \epsilon\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

One way to summarize this Hopf algbera $\mathcal{O}(SL_2)$ is as the freely generated algbera $k[a,b,c,d]/\mathcal{I}$ where I is the biideal generated by the element $ad-bc-1$, so:

$$\mathcal{I} = \langle ad-bc-1\rangle_{\mathcal{O}\langle SL_2\rangle}$$

and one interesting feature is that the determinant function $\det \in \mathcal{O}(SL_2)$ turns out to be a group-like element, in fact:

$$\begin{aligned}\Delta(ad-bc) &= \Delta(a)\Delta(d) - \Delta(b)\Delta(c)\\ &= (a\otimes a + b\otimes c)(c\otimes b + d\otimes d) - (a\otimes a + b\otimes d)(c\otimes a + d\otimes c)\\ &= ac\otimes ab + bc\otimes cb + ad\otimes ad + bd\otimes cd - bc\otimes da - bd\otimes dc\\ &\quad - ac\otimes ba - ad\otimes bc\\ &= ad\otimes(ad-bc) - bc\otimes(da-cb)\\ &= (ad-bc)\otimes(ad-bc).\end{aligned}$$

and hence it is a cocentral (meaning it cocommutes).

In the interest of continuity, the next example serves as a natural stepping stone for the next chapter by exploring a latent duality within its construction. Observe that for each $g \in SL_2(k)$ we can associate an element of $\mathrm{Alg}_k(\mathcal{O}(SL_2),k)$ (the Hopf algbera of k-algebra maps from $\mathcal{O}(SL_2)$ to k). Let us call this identification:

$$\begin{aligned}\Psi\colon SL_2(k) &\longrightarrow \mathrm{Alg}_k(\mathcal{O}(SL_2),k)\\ g &\longmapsto \Psi(g)(\varphi) := \langle\varphi, g\rangle = \varphi(g)\end{aligned}$$

this should remind us of identifying a space with its double dual because it really is just that. Given that these are both finite-dimensional vector spaces Ψ really is a bijective map, moreover it is also a Hopf algebra morphism, we won't check this here, it would first require equipping $\mathcal{O}(SL_2)$ with a Hopf structure and then checking that all structures are preserved along Ψ. The former is inherited naturally from the domain of its maps and the latter is a typical calculation in abstract nonsense. Lets instead just remark that, for

one, it is well-defined and we check that by ensuring that Ψ respects the quotient structure of $\mathcal{O}(SL_2)$ i.e., the determinant 1 condition:

$$\Psi(g)(a) = \langle a, g\rangle = g_{11} \qquad \Psi(g)(b) = \langle b, g\rangle = g_{12}$$
$$\Psi(g)(c) = \langle c, g\rangle = g_{21} \qquad \Psi(g)(d) = \langle d, g\rangle = g_{22}$$

$$\Psi(g)(ad - bc) = \langle ad - bc, g\rangle = g_{11}g_{22} - g_{12}g_{21} = \det(g) = 1$$

The next chapter is intent on testing the axioms for Hopf algebras by looking for symmetry among them. Hopefully it will increase one's understanding of the way in which Hopf algebras can be considered as an improvement on the way we discuss symmetry. In part, by noticing that they capture within their diagrammatic structuring all group theory and Lie algebra theory (as we have seen), subjects which have long been considered to be the universal currencies for meaningful discussions of symmetry in geometry, in algebra, in differential equations, in physics you name it.

8 Duality and *-structures

8.1 Duality

In this section we explore in detail the theme with which we ended on the last chapter: Duality. One of the remarkable things about all the axioms we provided for bialgebras, and is certainly also retained by Hopf algebras, is that if we decided to construct a co-Hopf algebra by reversing all arrows, as we did in the beginning to get the coalgberas from the algebras for instance, we will get after relabelling, and the reader can easily test this by "manually" forcing this process on the diagrams below, the precise same axioms. It is as if we had an equilateral triangle with labeled vertices, and after rotating it by some multiple of $\pi/3$ and then relabeled the vertices appropriately we get an exactly looking triangle. It is essentially telling us that the we should when working with Hopf algebras, treat as democratically and equitably as possible, the notions of algebra and coalgebra.

There is more than one way to think about or make sense of this symmetry (duality), and throughout the rest of the text it may be the case that it manifests itself as the omniscient power shaping our ideas without objectively imposing them. The next paragraphs and results though, will be focusing exclusively on the duality that is most evident, which is the one provided by the dual of the underlying vector space of the (Hopf) bialgebra.

Lets say that $(A, m, \Delta, \eta, \epsilon, S)$ is some Hopf algebra, we have already said in Chapter 1 that if A is a finite-dimensional vector space we can, by relabelling the adjoints of our structural maps as their (co)-counterpart, construct a Hopf algebra $(A^*, \Delta^*, m^*, \eta^*, S^*)$ and since $A^{**} \cong A$ we should make sure that from here on we think of A and A^* as

symmetric objects. For that purpose we should introduce the notation suggested in the last chapter and that is used in almost all books on related-subjects: the pairing notation $\langle -, - \rangle$ for the evaluation map $\mathrm{ev}\colon V^* \otimes V \longrightarrow k$. We employ it because, barring its inherent choice for who sits on the left and who sits on the right, it is still substantially more "democratic" in treating both objects than the notation for evaluation, $\omega(v)$ with $\omega \in V^*, v \in V$, that we typically use.

We can now rephrase, in an arguably more equitable sense, how we define the operations on A^* from adjointeness of the ones on A. Lets first look at the product $\cdot := \Delta^*$ for dual vectors:

$$\langle \alpha \cdot \beta, v \rangle := \langle \alpha \otimes \beta, \Delta(v) \rangle$$

remember here that $\Delta(v) = \sum v_{(1)} \otimes v_{(2)}$. This product has to be associative since we know its dual, the coproduct, is coassociative (duality at play, one could not ask for one without the other). In fact:

$$\begin{aligned}
\langle (\alpha \cdot \beta) \cdot \xi, v \rangle &= \langle (\alpha \cdot \beta) \otimes \xi, \Delta(v) \rangle \\
&= \sum \langle (\alpha \cdot \beta) \otimes \xi, v_{(1)} \otimes v_{(2)} \rangle \\
&= \sum \langle (\alpha \otimes \beta) \otimes \xi, \Delta(v_{(1)}) \otimes v_{(2)} \rangle \\
&= \sum \langle (\alpha \otimes \beta) \otimes \xi, v_{(1)} \otimes \Delta(v_{(2)}) \rangle \\
&= \sum \langle \alpha \otimes (\beta \otimes \xi), v_{(1)} \otimes \Delta(v_{(2)}) \rangle \\
&= \sum \langle \alpha \otimes (\beta \cdot \xi), v_{(1)} \otimes v_{(2)} \rangle \\
&= \sum \langle \alpha \otimes (\beta \cdot \xi), \Delta(v) \rangle \\
&= \langle \alpha \cdot (\beta \cdot \xi), v \rangle
\end{aligned}$$

The first equality is given as the definition, the second is just a translation to Sweedler notation, the third is reapplying he definition, the fourth is coassociating the coproduct, the fifth is associativity of the tensor product, and the remaining ones are a strict mirroring of this process. In the end we can establish that in fact: $(\alpha \cdot \beta) \cdot \xi = \alpha \cdot (\beta \cdot \xi)$, our newly defined multiplication is indeed associative. We will not prove the properties of all these things but let us define what still needs to be defined starting with the dual coproduct:

$$\langle \Delta(\alpha), v \otimes w \rangle := \langle \alpha, vw \rangle$$

its coassociativy is also guaranteed through the associativity of the product and the proof is very similar to the one given above. The unit counit are given by:

$$\begin{aligned}
\langle 1, v \rangle &:= \epsilon(v) \\
\epsilon(\alpha) &:= \langle \alpha, 1 \rangle
\end{aligned}$$

here the notation makes it almost hard to distinguish between both cases and for good reason since we want to treat them as equal as possible. To finish, the antipode:

$$\langle S(\alpha), v\rangle := \langle \alpha, S(v)\rangle$$

lets check it is in fact the antipode for one of the equations since the other one is completely analogous:

$$\begin{aligned}
\langle S(\alpha_{(1)})\cdot\alpha_{(2)}, v\rangle &= \langle S(\alpha_{(1)})\otimes\alpha_{(2)}, \Delta(v)\rangle \\
&= \langle \alpha_{(1)}\otimes\alpha_{(2)}, S(v_{(1)})\otimes v_{(2)}\rangle \\
&= \langle \Delta(\alpha), S(v_{(1)})\otimes v_{(2)}\rangle \\
&= \langle \alpha, S(v_{(1)})v_{(2)}\rangle \\
&= \langle \alpha, \epsilon(v)1\rangle \\
&= \epsilon(v)\epsilon(\alpha) \\
&= \langle \epsilon(\alpha)1, v\rangle \quad \Longrightarrow (S\otimes \mathrm{id})(\Delta(\alpha)) = \epsilon(\alpha)1
\end{aligned}$$

we will then say that any two Hopf algebras H, A for which we can define a pairing i.e, a k-bilinear map $\langle -,-\rangle : H\otimes A \longrightarrow k$ with the above properties are then said to be dually paired or strictly dually paired if the mapping is non-degenerate. The case we just built for the dual of a a finite-dimensional Hopf algebra is an example of a strict dual pair. Let us now see how some of our examples are dually paired.

Example 8.1. The group Hopf algebra kG for a finite group G, is dually paired with the Hopf algebra $k(G)$, meaning that $k(G) \cong (kG)^*$ or the other way around $kG \cong k(G)^*$.The isomorphism as Hopf algberas is naturally constructed by the dual pairing, so to prove its existence let us construct one explicitly:

$$\langle \alpha, \sum_{g\in G} a(g)g\rangle := \sum_{g\in G} a(g)\alpha(g); \qquad \forall \alpha \in k(G), \forall\Big(\sum_{g\in G} a(g)g\Big) \in kG$$

this is a non-degenerate mapping and we have that:

$$\begin{aligned}
\langle \alpha\beta, \sum_{g\in G} a(g)g\rangle &= \sum_{g\in G} a(g)\alpha\beta(g) = \sum_{g\in G} a(g)\alpha(g)\beta(g) \\
&= \sum_{g\in G} a(g)[\alpha\otimes\beta](g\otimes g) = \langle \alpha\otimes\beta, \sum_{g\in G} a(g)\Delta(g)\rangle \\
&= \langle \alpha\otimes\beta, \Delta\Big(\sum_{g\in G} a(g)g\Big)\rangle; \\
\langle \Delta(\alpha), \sum_{g,h\in G} a(g)b(h)(g\otimes h)\rangle &= \sum_{g,h\in G} a(g)\underline{(h)}\Delta(\alpha)(g\otimes h) = \sum_{g,h\in G} a(g)b(h)\alpha(gh) \\
&= \langle \alpha, \sum_{g,h\in G} a(b)b(h)gh\rangle = \langle \alpha, \Big(\sum_{g\in G} a(b)g\Big)\Big(\sum_{h\in G} b(h)h\Big)\rangle \\
\langle S(\alpha), \sum_{g\in G} a(g)g\rangle &= \sum_{g\in G} a(g)S(\alpha)(g) = \sum_{g\in G} a(g)\alpha(g^{-1}) \\
&= \sum_{g\in G} a(g)S(\alpha)(g) = \langle \alpha, \sum_{g\in G} a(g)g^{-1}\rangle \\
&= \langle \alpha, S\Big(\sum_{g\in G} a(g)g\Big)\rangle
\end{aligned}$$

we leave out the counit and unit ones given their simplicity.

The next case is of $U(\mathfrak{g})$ and $\mathcal{O}(G)$ when $\mathfrak{g}$ is the finite-dimensional semisimple complex Lie algebra corresponding to the Lie group G, we won't explore this in detail and leave it alone by saying that if ρ is the defining representation of $\rho\colon \mathfrak{g} \longrightarrow M_{\dim(\mathfrak{g})}(\mathbb{C})$ then the pairing is defined as:

$$\langle X, x^i_j\rangle := \rho(X)_{ij}$$

where $X \in U(\mathfrak{g})$ and $x^i_j \in \mathcal{O}(G)$.

8.2 *-structures

9 Actions and Coactions

In this chapter we treat the diverse theory of representations or modules that associated to Hopf algebras. Some of the challenge come from the way it requires bits of brain-flipping language to fully capture them (a clear consequence of the duality). Once again we will mirror-reflect all of the ideas and constructions for actions and modules in the second subchapter when we discuss coactions or comodules.

At the end of the chapter we summarize, through canonical examples, most of what will have been abstractly constructed up to that point, this will serve itself extremely

valuable when we attempt to define the notion of bicrossed products, quantum doubles of Hopf algebras and quantum doubles as conceived by Kitaev

9.1 Modules

It is a known and important fact, that when working with representations or modules of some fixed finite group G, we can construct the following modules:

- A trivial action or module k such that $g \triangleright \lambda = \lambda, \quad \forall g \in G, \quad \forall \lambda \in k^{\times}$;
- Given two representations (V, π) and (W, π') we can form a representation by adding them via the direct product:

$$\begin{aligned} \pi \oplus \pi' \colon G &\longrightarrow \mathrm{GL}(V \oplus W) \\ g &\longmapsto \pi_g \oplus \pi'_g(v, w) := (\pi_g(v), \pi'_g(w)) \end{aligned}$$

  and almost as important we can form a representation by multiplying them via the the tensor product:

$$\begin{aligned} \pi \oplus \pi' \colon G &\longrightarrow \mathrm{GL}(V \otimes W) \\ g &\longmapsto \pi_g \otimes \pi'_g(v \otimes w) := \pi_g(v) \otimes \pi'_g(w); \end{aligned}$$

- We can also associate to any (V, π) a dual representation (V^*, ψ) not, as one could naively guess, by considering the adjoints $\pi_g^* \colon V^* \longrightarrow V^*$ since then π^* would then be a group antihomomorphism given the contravariant nature of the dualization. Proof of concept, if we have $\pi_{gh} = \pi_g \circ \pi_h$ then the adjoint behaves as:

$$\pi^*_{gh} = (\pi_g \circ \pi_h)^* = \pi^*_h \circ \pi^*_g \neq \pi^*_g \circ \pi^*_h$$

  unless the group is abelian of course, but it would fail in general. So we can associate the following dual representation:

$$\psi_g := (\pi_{g^{-1}})^*$$

  one can check that in fact:

$$\psi_{gh} = (\pi_{h^{-1}g^{-1}})^* = (\pi_{h^{-1}} \circ \pi_{g^{-1}})^* = (\pi_{g^{-1}})^* \circ (\pi_{g^{-1}})^* = \psi_g \circ \psi_h$$

  it is a group acting (covariantly) on V^*.

If one has had experience with this type of theory then one knows that for algebras we cannot, in general, do any of the above, with the exception of direct sums, without requiring additional structures. Before proceeding lets recall the definition for a representation of an algebra [2] A :

[2] we change our notation slightly here by introducing $\triangleright$

Definition 9.1. We say that $(V, \triangleright)$, where V is a vector space over k, is a left action or left representation of an algebra A if $\triangleright\colon A \otimes V \longrightarrow V$ is a k-linear map satisfying the following (where we define $\triangleright(a \otimes v) := a \triangleright v$):

$$\begin{aligned} ab \triangleright v &= (a \triangleright v)(b \triangleright v), \quad \forall a, b \in A, \ \forall v \in V \\ 1 \triangleright v &= v \quad \forall a \in A, \ \forall v \in V \end{aligned}$$

an equivalent way to rephrase this is by saying that V is a left A-module. Here the meaning of left comes from the fact that we think of the action as being performed from the left, since we place A at the left factor of the tensor product, and there is nothing preventing us from putting it on the right. Those we will call right actions of A or right A-modules. Another way of thinking about this definition, and often the one that shows up first, is to think of $\triangleright$ by fixing the first entry of the tensor product. Let $a \in A$:

$$\begin{aligned} \triangleright(a \otimes -) : V &\longrightarrow V \\ v &\longmapsto a \triangleright v \end{aligned}$$

is a linear map $\in \mathrm{End}(V)$ which defines the following:

$$\begin{aligned} \rho\colon A &\longrightarrow \mathrm{End}(V) \\ a &\longmapsto \rho(a) := \triangleright(a \otimes -) \end{aligned}$$

satisfying:

$$\rho(ab) = \triangleright(ab \otimes -) = \triangleright(a \otimes -) \circ \triangleright(v \otimes -) = \rho(a) \circ \rho(b)$$

and so we can alternatively define an action of an algebra A on V by constructing an algebra homomorphism ρ between A and the algebra $\mathrm{End}(V)$ of k-linear maps from V to V.[3]

Definition 9.2. Let a H be a Hopf algebra, we say that a k-linear map $psi\colon V \longrightarrow W$ between two left H-modules is a left H-module homomorphism or left H-intertwiner if:

$$\psi(h \triangleright v) = h \triangleright \psi(w)$$

When we have $\phi\colon H \longrightarrow A$ a Hopf algebra homomorphism and a left -module V, we can induce in V an A-module structure by pulling-back the action via ϕ:

$$H \xrightarrow{\phi} A \xrightarrow{\rho} \mathrm{End}(V)$$

Just by looking at this definition we can begin puzzling over what is the trivial representation of A? It certainly can't be, as is the case for groups, just $\rho_{triv}(a) = \mathrm{id}_k$ for all

[3] remember this is an algebra for the composition and with unit given by the identity map

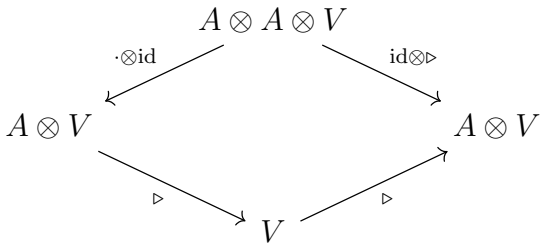


Figure 5: Diagrams for an A-module V.

$a \in A$, quite simply because this is not an algebra homomorphism. So we don't necessarily have enough structure in our algebra for defining this ρ_{triv}. All that we really know is that that it should be an algebra homomorphism $A \longrightarrow k = \mathrm{End}(k)$. The notion of a counit ϵ, unique as it is in a bialgebra, fulfills precisely that purpose and allows us to define the following trivial representation $a \triangleright \lambda := \epsilon(a)\lambda \in k$.

Now remain two of our three original concerns, namely the possibility of multiplying algebra representations by applying the tensor product. We begin by puzzling about how can we combine two representations acting on V and W in a way they can act on $V \otimes W$? The answer should provide an algebra map that can split $a \in A$ into a linear sum of tensor factors $\in A$ so that we can act on $v \otimes w \in V \otimes W$ independently with our two original representations and this is the precise structure introduced by the coproduct $\Delta : A \longrightarrow A \otimes A$. One way to think about it is to note that given a pair of left A-modules V and W we can define define a left $A \otimes A$-module bu tensoring their actions. In order for us to define $V \otimes W$ as an A-module we need to have an algebra map $\Delta : A \longrightarrow A \otimes A$ which we can consistently deploy to pull-back the tensored actions of V and W.

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho \otimes \rho} \mathrm{End}(V \otimes W)$$

We are quickly realizing that the structures we have been adding to our algebras have quite satisfactory categories of representation. For bialgebras we have that $\mathrm{Rep}(A)$ is a braided tensor category.

We can now restrict the definition to include A-modules that are not only vector spaces but also algebras and in doing so we need to include provisions for the axioms that respect the underlying structure. We introduce the new definition then:

Definition 9.3. Let H be a Hopf algebra, a left H-module algebra A is vector space over k and :

- A is a left H-module for the action $\triangleright$;
- the multiplication $m \colon A \otimes A \longrightarrow A$ commutes with the action, or in other words:

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b);$$

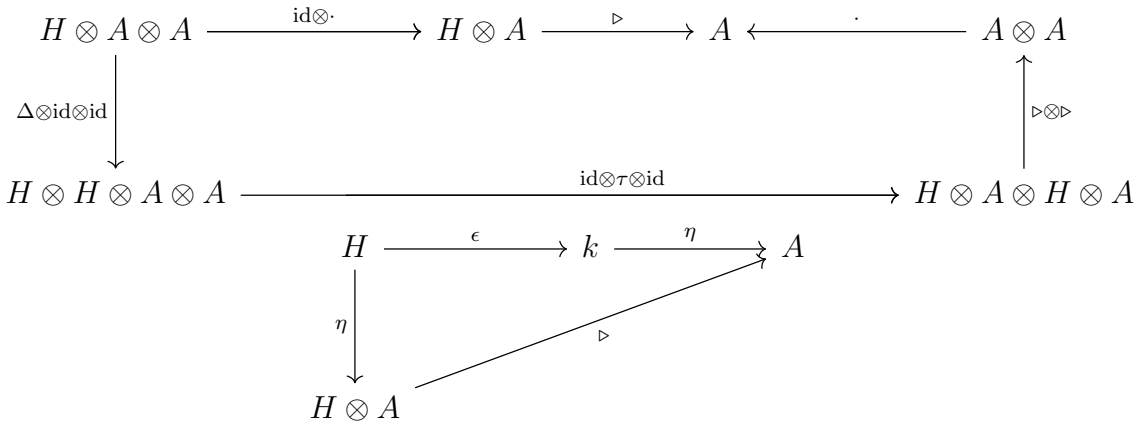


Figure 6: Axioms for an H-module algebra A

- the unit map $\eta\colon A \longrightarrow k$ commutes with the action, or in other words:

$$h \triangleright 1 = \epsilon(h)1$$

Example 9.1. Given some Hopf algebra H we can consider it as a left H-module algebra by the left adjoint action Ad. The action is given by:

$$h \triangleright g = \mathrm{Ad}_h(g) := \sum h_{(1)} g S(h_{(2)})$$

It is a left H-module since:

$$\begin{aligned} h \triangleright (g \triangleright a) &= h \triangleright (\mathrm{Ad}_g(a) = h \triangleright \Big(\sum g_{(1)} a S(g_{(2)}) \Big) \\ &= \sum h_{(1)} g_{(1)} a S(g_{(2)}) S(h_{(2)}) \\ &= \sum (hg)_{(1)} a S((hg)_{(2)}) \\ &= \mathrm{Ad}_{hg}(a) \\ &= hg \triangleright a \end{aligned}$$

it is a left H-module algebra because it also satisfies:

$$\begin{aligned} h \triangleright (ga) = \mathrm{Ad}_h(ga) &= \sum h_{(1)} g a S(h_{(2)}) = \sum h_{(1)} g S(h_{(2)}) h_{(3)} a S(h_{(2)}) \\ &= \mathrm{Ad}_{h_{(1)}}(g) \mathrm{Ad}_{h_{(2)}}(h) \\ &= (h_{(1)} \triangleright g)(h_{(2)} \triangleright a) \end{aligned}$$

and

$$h \triangleright 1 = \mathrm{Ad}_h(1) = \sum h_{(1)} S(h_{(2)}) = \epsilon(h)1$$

For the cases of $H = kG$ and $H = U(\mathfrak{g})$, we get that, if A is a left-module algebra than the adjoint action is given by:

$$g \triangleright a = \sum g_{(1)} a g_{(2)} = g a g^{-1}$$
$$X \triangleright a = \sum X_{(1)} a X_{(2)} = Xa - aX = [X, a]$$

respectively. This corresponds to saying that G acts by automorphisms and that $\mathfrak{g}$ acts by derivations on A For the group case we would say that the algebra is G-covariant and for the Lie algebra case we would say it is $\mathfrak{g}$-covariant therefore the left adjoint H-module is a generalized construction for Hopf algebras.

Example 9.2. When we have an Hopf algebra A dually paired to H we have a left A-module algebra called coregular left-module algebra, given by:

$$\alpha \triangleright h := R^*_\alpha(h) = \sum h_{(1)} \langle \alpha, h_{(2)} \rangle$$

First we check it is in fact a left A-module:

$$\begin{aligned} \alpha \triangleright (\beta \triangleright h) &= R^*_\alpha\Big(\sum h_{(1)} \langle \beta, h_{(2)} \rangle \Big) = \sum h_{(1)} \langle \alpha, h_{(2)} \rangle \langle \beta, h_{(3)} \rangle \\ &= \sum h_{(1)} \langle \alpha \otimes \beta, h_{(2)} \otimes h_{(3)} \rangle = \sum h_{(1)} \langle \alpha \otimes \beta, \Delta(h_{(2)}) \rangle \\ &= \sum h_{(1)} \langle \alpha \cdot \beta, h_{(2)} \rangle = R^*_{\alpha \cdot \beta}(h) = (\alpha \cdot \beta) \triangleright h \end{aligned}$$

to guarantee that it is left A-module algebra it must satisfy:

$$\begin{aligned} \alpha \triangleright (hg) &= R^*_\alpha(hg) = \sum (hg)_{(1)} \langle \alpha, (hg)_{(2)} \rangle = \sum h_{(1)} g_{(1)} \langle \alpha, h_{(1)} g_{(2)} \rangle \\ &= \sum h_{(1)} g_{(1)} \langle \Delta(\alpha), h_{(2)} \otimes g_{(2)} \rangle = \sum h_{(1)} \langle \alpha_{(1)}, h_{(2)} \rangle g_{(1)} \langle \alpha_{(2)}, g_{(2)} \rangle \\ &= (\alpha_{(1)} \triangleright h)(\alpha_{(2)} \triangleright g) \end{aligned}$$

and also:

$$\alpha \triangleright 1 = \langle 1, \alpha \rangle 1 = \epsilon(h)$$

These constructions motivate the following proposition/definition:

Definition 9.4. Let H be a Hopf algebra and A a left H-module algebra. We say that the algebra defined on $A \otimes H$ by:

$$(a \otimes h)(b \otimes g) = \sum a(h_{(1)} \triangleright b) \otimes (h_{(2)} g)$$

with the unit element $1 \otimes 1$ is called the left cross product algebra $A \rtimes H$.

Proof. Associativity of the product comes from the following:

$$\begin{aligned}(a \otimes h)((b \otimes g)(c \otimes f)) &= \sum (a \otimes h) \otimes \Big(b(g_{(1)} \triangleright c) \otimes (g_{(2)} f)\Big) \\ &= \sum a(h_{(1)} \triangleright (b(g_{(1)} \triangleright c)) \otimes (h_{(2)} g_{(2)} f) \\ &= \sum a((h_{(1)})_{(1)} \triangleright b)((h_{(1)})_{(2)} g_{(1)} \triangleright c)) \otimes (h_{(2)} g_{(2)} f) \\ &= \sum a(h_{(1)} \triangleright b)(h_{(2)} g_{(1)} \triangleright c)) \otimes (h_{(3)} g_{(2)} f) \\ &= \sum \big(a(h_{(1)} \triangleright b) \otimes (h_{(2)} g)\big)(c \otimes f) \\ &= ((a \otimes h)(b \otimes g))(c \otimes f)\end{aligned}$$

and the unit element is $1 \otimes 1$ since:

$$(1 \otimes 1)(a \otimes b) = 1(1 \triangleright a) \otimes (1h) = a \otimes h$$

We can think of A and H as subalgberas of $A \rtimes H$ by the embedding provided by $a \hookrightarrow a \otimes 1$ and $h \hookrightarrow 1 \otimes h$ respectively. We can therefore write $ah := (a \otimes 1)(1 \otimes h) = a \otimes h$ and $ha := (1 \otimes h)(a \otimes 1) = \sum (h_{(1)} \triangleright a) \otimes h_{(2)}$ which we can identify with $\sum (h_{(1)} \triangleright a) h_{(2)}$. □

If we dualize this definition to left actions on coalgberas we get:

Definition 9.5. Let H be a Hopf algebra, a left A-module coalgebra C is vector space over k and :

- C is a left H-module for the action $\triangleright$;
- the comultiplication $\Delta \colon C \longrightarrow C \otimes C$ commutes with the action, or in other words:

$$\Delta(h \triangleright c) = \sum (h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleright c_{(2)});$$

- the counit map $\eta \colon A \longrightarrow k$ commutes with the action, or in other words:

$$\epsilon(h \triangleright c) = \epsilon(h)\epsilon(c)$$

these axioms are summarized by the commutative diagrams in Figure 5.

Example 9.3. The simple left regular action $L_h(g) = gh$ of a Hopf algebra acting on itsef is an example of a left H-module coalgbera. To verify we begin making sure that the action is in fact a left module:

$$\begin{aligned}h \triangleright (g \triangleright a) := L_h(L_g(a)) = (ag)h = a(gh) \\ = L_{gh}(a) = hg \triangleright a\end{aligned}$$

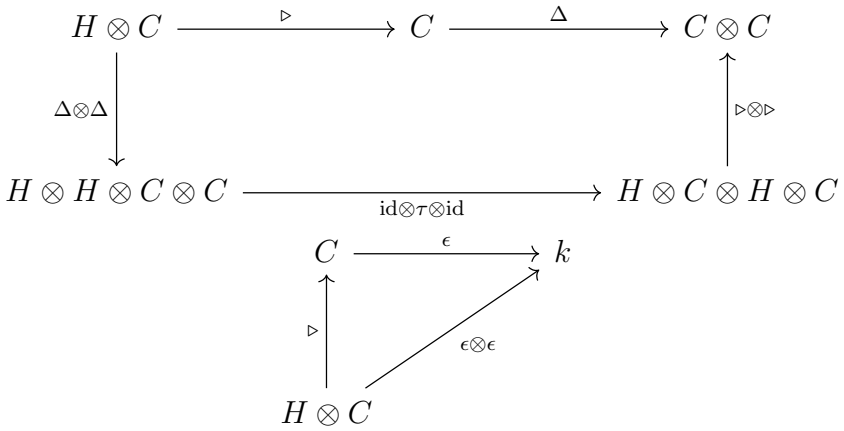


Figure 7: Axioms for a left H-module coalgbera

to conclude the proof of the statement let us check that Δ commutes with the action:

$$\begin{aligned}\Delta(h \triangleright g) &:= \Delta(L_h(g)) = \Delta(hg) = \Delta(h)\Delta(g) \\ &= \sum h_{(1)}g_{(1)} \otimes h_{(2)}g_{(2)}\end{aligned}$$

and that the same goes for the counit:

$$\epsilon(h \triangleright g) := \epsilon(hg) = \epsilon(h)\epsilon(g)$$

All of the constructions we have come across in this subsection have natural opposite constructions. Namely, for every time we have mentioned the word left we have "left" open the possibility of considering the right(-handed) analogue instead.

Let us define then the notion of right H-module by purely mirroring the diagrams of Figure 4. and we will write $v \triangleleft a := \triangleleft(v \otimes h)$ where $\triangleleft \colon V \otimes H \longrightarrow V$ is defined by the "vertically-reflected" axioms of a left module. Meaning it satisfies:

$$(v \triangleleft h) \triangleleft g = v \triangleleft hg, \quad (v \triangleleft 1) = v \qquad \forall g, h \in A, \forall v \in V$$

Both notions of left-module algebra and left-module coalgebra have a respective right-module algebra and right-module coalgebra counterpart. They can be obtained by the same "vertical-reflection" operation on the axioms and are summarized by the following equations:

$$\begin{aligned}(ab) \triangleleft h &= \sum (a \triangleleft h_{(1)})(b \triangleleft h_{(2)}), \quad (1 \triangleleft h) = \epsilon(h)1 \qquad \forall h, \forall a, b \in A \\ \Delta(c \triangleleft h) &= \sum (a_{(1)} \triangleleft h_{(1)})(a_{(2)} \triangleleft h_{(2)}), \quad \epsilon(c \triangleleft h) = \epsilon(c)\epsilon(h), \qquad \forall h, \forall c \in C\end{aligned}$$

where A is a right H-module algebra and C a right H-module coalgebra. The antipode S here provides a natural bridge between left-modules and right-modules since we can,

given a left Hopf algebra H-module, define a right H-module by $a \triangleleft h := S(h) \triangleright a$. The antialgebra homomorphism property of the antipode guarantees all the pre-requisites for a right action. It is simple to prove that a similar connection can be built from right to left modules when S^{-1} exists.

We end the subsection by defining a right cross product algebra just as we did for left-module algebras.

Proposition 9.1. *Let H be a Hopf algebra and A a right H-module algebra. We say that the algebra defined on $H \otimes A$ by:*

$$(a \otimes h)(g \otimes b) := \sum hg_{(1)} \otimes (a \triangleleft g_{(2)})b$$

with the unit element $1 \otimes 1$ is called the right cross product algebra $H \triangleright < A$.

Proof. The proof of this proposition is completely analogous to the left case that we have already proved. □

9.2 Comodules

The notion of an H-comodule, as one would expect, is obtained from the notion of H-module by reversing all arrows in its definition. We begin, by opposition to the last subsection, by defining right H-comodules.

Definition 9.6. A pair (β, V), where v is a k-vector space and $\beta : V \longrightarrow V \otimes H$ is said to be a right H-comodule or corepresentation or coaction if it satisfies the following equations:

$$(\otimes\Delta) \circ \beta = (\beta\otimes) \circ \beta \tag{17}$$

$$(\otimes\epsilon) \circ \beta = \tag{18}$$

its tacit now how to define a left H-comodule. Additional notation, naturally inspired by Sweedler's, is needed here, namely we need an equivalent of $\triangleright$ for coaction. The notation is then defined as follows:

$$\beta(v) = \sum v^{(\bar{1})} \otimes v^{(\bar{2})}$$

allowing us to re-write the axioms for a right coaction as:

$$\sum v^{(\bar{1})} \otimes (v^{(\bar{2})})_{(1)} \otimes (v^{(\bar{2})})_{(2)} = \sum v^{(\bar{1})(\bar{1})} \otimes v^{(\bar{1})(\bar{2})} \otimes v^{(\bar{2})}$$

$$\sum v^{(\bar{1})}\epsilon(v^{(\bar{2})}) = v$$

since this is a completely mirrored view of what happened for modules it is only natural that a coalgbera map, rather than pulling back actions of modules, "pushes out" a a coaction of comodules. In simple terms, if $\varphi \colon C \longrightarrow D$ is a coalgbera map, and we have a

coaction of C we can use φ to "transmit" it or to "push it out" to a coaction of D. We can also easily take the previous definitions for left and right H-module algebra and coalgebra and, just by virtue of reversing arrows, get definitons for left and right H-comodule coalgbera and algebra. Succinctly, and only for the right-handed case, we have that a A is an H-comodule algebra if:

$$\beta(ab) = \beta(a)\beta(b), \quad \beta(1) = 1 \otimes 1, \quad \forall a, b \in A$$

where we have on $V \otimes A$ the natural tensor algebra structure. In other words, the multiplication $m\colon A \otimes A \longrightarrow A$ and unit $\eta\colon k \longrightarrow A$ are H-comodule intertwiners. For C a right H-comodule coalgebra case the axioms look like:

$$\sum (c^{(\bar{1})})_{(1)} \otimes (c^{(\bar{1})})_{(2)} \otimes c^{(\bar{2})} = \sum (c_{(1)})^{(\bar{1})} \otimes (c_{(1)})^{(\bar{2})} \otimes (c_{(1)})^{(\bar{2})} (c_{(2)})^{(\bar{2})}$$
$$\sum \epsilon(c^{(\bar{1})}) c^{(\bar{2})} = \epsilon(c)$$

In the finite-dimensional case we can, since if H is a bialgebra then H^* is also a bialgebra, elaborate a way to go from V right H^*-comodule to V a left H-module by defining $h \triangleright v := \sum v^{(\bar{1})} \langle v^{(\bar{2})}, h \rangle$, where $v, v^{(\bar{1})} \in V$ and $v^{(\bar{2})} \in H^*$. Notice that:

$$\begin{aligned} h \triangleright (g \triangleright v) &= \sum (g \triangleright v)^{(\bar{1})} \langle (g \triangleright v)^{(\bar{2})}, h \rangle = \sum v^{(\bar{1})(\bar{1})} \langle v^{(\bar{1})(\bar{2})}, h \rangle \langle v^{(\bar{2})}, g \rangle \\ &= v^{(\bar{1})(\bar{1})} \langle v^{(\bar{1})(\bar{2})} \otimes v^{(\bar{2})}, g \otimes h \rangle = v^{(\bar{1})} \langle (v^{(\bar{2})})_{(1)} \otimes (v^{(\bar{2})})_{(2)}, g \otimes h \rangle \\ &= v^{(\bar{1})} \langle v^{(\bar{2})}, gh \rangle = (gh) \triangleright v \end{aligned}$$

where we used the duality properties we defined last chapter on the penultimate equality and the axiom for a right comodules on the third equality. Additionally, if C is a right H^*-comodule algebra/coalgebra then it can be turned into C a left H-module algebra/coalgebra. The last statement is instantly verified by our proof of the correspondence.

Now we finish by constructing the dual versions for left and right cross product algebras. We build, by using comodules instead, a notion of left and right cross coproduct coalgebras.

Definition 9.7. If H a bialgbera or Hopf algbera and C is a right H-comodule coalgebra, we can define on $H \otimes C$ a coalgebra structure via:

$$\Delta(h \otimes c) = h_{(1)} \otimes (c_{(1)})^{(\bar{1})} \otimes h_{(2)} (c_{(1)})^{(\bar{2})} \otimes c_{(2)}, \qquad \epsilon(h \otimes c) = \epsilon(h)\epsilon(c)$$

$\forall h \in H$ and $\forall c \in C$. We call this the right cross coproduct coalgbera $H \blacktriangleright\!< C$.

Let us check this is a well-defined coalgebra:

10 The Yang-Baxter Equation

Before plunging into the world of braided Hopf algebras also known as quasitriangular Hopf algebras it is instructive to begin by an attempt at explaining why the Yang-Baxter Equation is an important source of research problems in both pure mathematics and theoretical physics.

The first example comes from the active field of Quantum Computation, more precisely the topological kind, where the braiding statistics of anyons dictate what kind of computations can be performed. Specifically quantum gates, the physical processes that carry out the computation, must correspond to unitary representations of the braid group

11 Quasitriangular Structures

In this section we will face the task of introducing an additional piece of machinery into our Hopf algebras. The idea is simple in its definition and albeit slightly weird-looking at first, it provides us with a sort of controlled form of non-cocommutativity, or in its dual form non-commutativity, that will allows us to prove results that are just slightly relaxed versions of results that we would have for cocoummutative or commutative cases.

Let us begin by looking at the definition for a quasitriangular Hopf algebra.

Definition 11.1. A quasitriangular bialgbera or Hopf algebra is pair (H, R) consisting of a bialgbera or Hopf algebra and an invertible element $R \otimes H$ with $R = \sum R^{(1)} \otimes R^{(2)}$ typically called universal R-matrix and satisfying the following conditions:

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12} \tag{19}$$

the cocommutativity control condition:

$$\tau(\Delta(h)) = R\Delta(h)R^{-1} \tag{20}$$

and we will always write:

$$R_{ij} = \sum 1 \otimes \cdots \otimes R^{(1)} \otimes \cdots \otimes R^{(2)} \otimes \cdots \otimes 1$$

$R^{(1)}$ and $R^{(2)}$ are the only non-unital elements and are in the i-th and j-th factors of the sum respectively. This is just a notation for the embedding of R in higher tensor powers of H. We will mostly refer to:

$$R_{12} = (R \otimes 1), \quad R_{23} = (1 \otimes R), \quad R_{13} = (\tau_{H\otimes H}\otimes)(1 \otimes R)$$

The next step is through the most important properties of this definition, but before getting into abstract computations it is of use to first interpret these axioms using our notations so far. Preserving the order by which they were presented, this is how they look (sum signs will be omitted here):

- $R^{(1)}_{(1)} \otimes R^{(1)}_{(2)} \otimes R^{(2)} = R^{(1)} \otimes R'^{(1)} \otimes R^{(2)} R'^{(2)}$;
- $R^{(1)} \otimes R^{(2)}_{(1)} \otimes R^{(2)}_{(2)} = R^{(1)} R'^{(1)} \otimes R'^{(2)} \otimes R^{(2)}$;
- $h_{(2)} R^{(1)} \otimes h_{(1)} R^{(2)} = R^{(1)} h_{(1)} \otimes R^{(2)} h_{(2)}$

So let us then begin by proving a structural result:

Proposition 11.1. *If (H, R) is a quasitriangular bialgebra then $R \otimes H$ satisfies:*

$$(\epsilon\otimes)(R) = (\otimes\epsilon)(R) = \tag{21}$$

and if H is Hopf then it additionally satisfies:

$$(S\otimes)(R) = R^{-1}, \quad (\otimes S)(R^{-1}) = R \tag{22}$$

implying that $(S \otimes S)(R) = R$

Proof. For the first equations we observe that:

$$\begin{aligned}(\epsilon \otimes \otimes)(\Delta\otimes)(R) = \epsilon(R^{(1)}_{(1)}) \otimes R^{(1)}_{(2)} \otimes R^{(2)} = 1 \otimes \epsilon(R^{(1)}_{(1)}) R^{(1)}_{(2)} \otimes R^{(2)} \\ = 1 \otimes R^{(1)} \otimes R^{(2)} = (1 \otimes R) = R_{23}\end{aligned}$$

while simultaneously, by the first axiom, it is also equal to:

$$\begin{aligned}(\epsilon \otimes \otimes)(R_{13} R_{23}) &= (\epsilon \otimes \otimes)(R^{(1)} \otimes R'^{(1)} \otimes R^{(2)} R'^{(2)}) \\ &= 1 \otimes R'^{(1)} \otimes \epsilon(R^{(1)}) R^{(2)} R'^{(2)} \\ &= (\epsilon\otimes)(R)(1 \otimes R) \\ &= (\epsilon\otimes)(R) R_{23} \\ &\Rightarrow (\epsilon\otimes)(R) = 1 \quad \text{since } R \text{ is invertible}\end{aligned}$$

the other equation is completely identical. For the Hopf algebra statements we have that:

$$(m\otimes)(\otimes S\otimes)(\Delta\otimes)(R) = R^{(1)}_{(1)} S(R^{(1)}_{(2)}) \otimes R^{(2)} = 1 \otimes 1$$

and also have that:

$$\begin{aligned}R(S\otimes)(R) = R^{(1)} S(R'^{(1)}) \otimes R^{(2)} R'^{(2)} = R^{(1)}_{(1)} S(R^{(1)}_{(2)}) \otimes R^{(2)} \\ = 1 \otimes 1 \Rightarrow (S\otimes)(R) = R^{-1}\end{aligned}$$

where we used the first axiom to get the third equality. The other side works similarly and the the last statement comes from combining both in an obvious way. If we examine condition (18) we get, by applying τ to both sides of the equation, $\Delta(h) =$

$\tau(R)\tau(\Delta(h))\tau(R^{-1}) = \tau(R)\Delta^{copp}(h))\tau(R^{-1})$ which implies that the coopposite Hopf algebra is also quasitriangular with antipode S^{-1} and $R_{21} := \tau(R^{-1})$ as the quasi-triangular element. This easily proves that $\tau \circ [(S^{-1}\otimes)(R_{21})] = (\otimes S^{-1})(R) = \tau(R_{21}^{-1}) = R^{-1}$ making use of the result we have just proved. Both results combine to give:

$$\begin{aligned}(S \otimes S)(R) &= (\otimes S)(S\otimes)(R) = (\otimes S)(R^{-1}) \\ &= (\otimes S)((\otimes S^{-1})(R)) = R\end{aligned}$$

□

Now we proceed to prove the most fundamental properties of quasi-triangular structures.

Proposition 11.2. *If (H, R) is a quasitriangular Hopf algbera, then the following equations holds:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Proof. A simple computation using the axioms shows that:

$$\begin{aligned}R_{12}R_{13}R_{23} &= R_{12}(\Delta\otimes)(R) = (R \otimes 1)(\Delta\otimes)(R) \\ &= ((\tau \circ \Delta)R\otimes)(R) = (\tau_{1,2}\otimes)(\Delta\otimes)(R)(R \otimes 1) \\ &= \tau_{1,2}(R_{13}R_{23})R_{12} = R_{23}R_{13}R_{12}\end{aligned}$$

we applied all three conditions that define a quasi-triangular structure. We applied the first condition of (17) in the first equality, condition (18) on the third equality and the second condition of (17) on the penultimate equation. □

A Graded Rings and Filtered Algebras

We say that R is a graded ring We say that a ring R is

B Lie Algebras

In this chapter we recall some of the basic and useful definitions associated to Lie algebras and enveloping algebras.

Definition B.1. A Lie algebra $\mathfrak{g}$ is a vector space over some field k, together with skew-symmetric and bilinear product $[-,-]$ satisfying the Jacobi identity:

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

For any k-algebra A we can associate a given Lie algebra $\mathbf{L}(A)$ with the bracket defined as $[x, y] := xy - yx$. This trivially satisfies the first two axioms, and the Jacobi identity comes from:

$$\begin{aligned}[[x, y], z] + [y, [x, z]] &= (xy - yx)z - z(xy - yx) + y(xz - zx) - (xz - zx)y \\ &= xyz - yxz - zxy + zyx + yxz - yzx - xzy + zxy \\ &= x(yz - zy) - (yz - zy)x = [x, [y, z]]\end{aligned}$$

This is the source of many examples for Lie algebras. We now look at its subobjects:

Definition B.2. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra if it satisfies: $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ and, in particular, a left or right ideal if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ or $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$, respectively, but since the product is skew-symmetric $[g, h] = -[h, g]$ there exist only two-sided ideals. Additionally, $\mathfrak{g}/\mathfrak{h}$ has an induced Lie bracket via the the canonical projection linear map $\pi\colon \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$. It is easy to see that $[x + \mathfrak{h}, y + \mathfrak{h}] := [x, y] + \mathfrak{h}$ is in fact a well defined Lie bracket well-defined.

Definition B.3. We say that a k-linear map $\varphi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$. The kernel of φ a homomorphism is an ideal since $\varphi([x, y]) = [\varphi(x), \varphi(y)] = 0$ if $x \in \ker(\mathfrak{g})$ for all $y \in \mathfrak{g}$ and the image of $\varphi(\mathfrak{g}) \subseteq \mathfrak{h}$ is a Lie subalgebra (an ideal if φ is surjective).

Example B.1. If V is an n-dimensional vector space over k, we define:

$$\mathbf{L}(\mathrm{End}(V)) \cong \mathbf{L}(M_n(k)) := \mathfrak{gl}(n)$$

as the Lie algebra of endomorphisms of V with the bracket defined as the commutator of operators. $\mathfrak{gl}(n) = \mathrm{Span}_k\{E_{11}, E_{12}, \ldots, E_{nn}\}$ where we define E_{ij} to be the matrix with a 1 on the i-th row and j-th column and 0 everywhere else. We can then write the commutator relations as:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{jk}$$

we define the subspace generated by the traceless matrices (for fields of characteristic $\neq 2$ as $\mathfrak{sl}(n)$. Since $(AB)_{ii} = \sum_k a_{ik}b_{ki}$ and $(BA)_{ii} = \sum_k b_{ik}a_{ki}$ we have that $\mathrm{Tr}(AB) = \sum_{j,k} a_{jk}b_{kj}$ and that $\mathrm{Tr}(BA) = \sum_{j,k} b_{jk}a_{kj} = \sum_{k,j} b_{kj}a_{jk} = \mathrm{Tr}(BA)$ for any two square matrices A, B . The implication of the preceding calculation being that since $\mathrm{Tr}([A, B]) = 0$ then $\mathfrak{sl}(2)$ is a subalgebra of $\mathfrak{gl}(n)$.

Definition B.4. The corresponding universal enveloping algebra to a Lie algebra $\mathfrak{g}$ which we deonte by the pair $\{U(\mathfrak{g}), \iota_{\mathfrak{g}}\colon \mathfrak{g} \longrightarrow \mathbf{L}(\mathfrak{g})\}$. Where

B.1 $\mathfrak{sl}(2)$ and $U(\mathfrak{sl}(2))$

We consider the three generating elements of $\mathfrak{sl}(2)$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and they satisfy the following commutation relations:

$$[X, Y] = H, \quad [H, X] = X, \quad [H, Y] = -Y$$